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THE OBSTACLE PROBLEM FOR MONGE-AMPÈRE TYPE EQUATIONS IN NON-CONVEX DOMAINS

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Abstract. In this paper, we consider the obstacle problem for Monge-Apmère type equations which include prescribed Gauss curvature equation as a special case. We establish $C^{1,1}$ regularity of the greatest viscosity solution in non-convex domains.

1. **Introduction.** Let Ω be a bounded domain in $\mathbb{R}^n$ with C^4 boundary $\partial\Omega$. Given a function $g \in C^3(\overline{\Omega})$, we shall concern the following obstacle problem

$$\begin{cases} \det D^2 u \geq \psi(x, u, Du) & \text{in } \Omega, \\ u \leq g & \text{in } \Omega, \\ u \text{ is locally convex in } \Omega, & \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $g \geq \varphi \in C^4(\partial\Omega)$, $\psi \in C^3(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\psi \geq 0$, $Du = (D_i u)$ and $D^2 u = (D_{ij} u)$ denotes the gradient and Hessian of u, respectively. We say u is *locally convex* in Ω, if u is convex in arbitrary ball $B_r(x) = \{y : |y - x| < r\} \subset \Omega$.

Denote $\mathcal{A} = \{u : u \text{ is a viscosity solution of (1.1) }\}$, see section 2 for the definition of viscosity solution. In the sequel, we may suppose the set $\mathcal{A}$ is nonempty. Then we would like to study the maximization problem

$$u(x) =: \sup_{v \in \mathcal{A}} v(x). \tag{P}$$

Our background is from finding the greatest hypersurface with an obstacle, whose Gauss-Kronecker curvature is bounded from below by a positive function. From the viewpoint of geometric applications, it is of interest to study the Dirichlet problem for Monge-Ampère equations in non-convex domains, See [9, 10, 11] and references therein.

Using Perron's method we show in the beginning of the next section

Theorem 1.1. *If $\mathcal{A}$ is nonempty, then the maximizer u of (P) is still in the class $\mathcal{A}$ and in viscosity sense*

$$\det D^2 u(x) = \psi(x, u(x), Du(x)), \ \ x \in \{x \in \Omega : u(x) < g(x)\}.$$

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In this paper we are interested in the regularity of the maximizer u of (P). When $\psi = 1, \varphi = 0$ and Ω is strictly convex, the problem (P) has been studied by Lee [13]. He proved the $C^{1,1}$ regularity of the viscosity solution and $C^{1,\alpha}$ regularity of free boundary. Another obstacle problem for Monge-Ampère equation was considered by Savin [16], he studied the minimum nonnegative function u satisfying $u = 1$ on $\partial\Omega$ and $Mu \le \mu_0$, where Mu is the Monge-Ampère measure (see [12]) of u. In [5], Caffarelli and McCann considered the free boundary problem of Monge-Ampère type equations related to optimal transportation problem.

We say $\psi(x, z, p)$ has *fine property*, if comparison principle holds for equation

$$\det D^2 u = \psi(x, u, Du) \quad \text{in } \Omega, \tag{1.2}$$

i.e., let u (resp. v) be a viscosity subsolution (resp. viscosity supersolution) to (1.2) and $u \le v$ on ∂G, then

$$u \le v \quad \text{in } G,$$

where $G \subset \Omega$ is an arbitrary domain.

Many functions have the fine property, such as $\psi = \psi(x) \ge 0$ and

$$\psi(x, z, p) = K(x)(1 + |p|^2)^{\frac{n+2}{2}},$$

where $K(x) \ge 0$, see [17].

The following theorem shows the regularity of the maximizer of (P).

Theorem 1.2. *Assume that $g > \varphi$ on Ω and $\psi > 0$ has fine property and there exists a function $\underline{u} \in \mathcal{A}$. If $\underline{u} \in C^2(\overline{\Omega})$, then the maximizer $u \in C^{1,1}(\overline{\Omega})$.*

After establish the $C^{1,1}$ regularity, the obstacle problem reduces to the obstacle problem for the uniformly elliptic equations.

Remark 1. In case $\psi \equiv \psi(x)$ or, more generally (due to P.L. Lions; see [6]), when ψ satisfies

$$0 \le \psi(x, z, p) \le C(1 + |p|^2)^{n/2} \quad \text{for } x \in \overline{\Omega},\ z \le \max \varphi,\ p \in \mathbb{R}^n,$$

one can construct a strictly convex subsolution $\underline{u} \in C^2(\overline{\Omega})$ to (1.2) with $\underline{u} = \varphi$ on $\partial\Omega$ if Ω is strictly convex; this fails for non-convex domains.

Our approach can be applied to obstacle problem for more general Monge-Ampère equations, see section 4 in the paper.

The paper is organized as follows. In section 2, we recall the definition of convex viscosity solution and prove Theorem 1.1. In section 3, we consider a singularity perturbation problem and prove Theorem 1.2. In section 4, we treat another Monge-Ampère type equation with obstacle.

2. **Existence and uniqueness of viscosity solution.** We open this section by recalling the notions of superjet and subjet and some facts for convex function. Then we use Perron's method to prove Theorem 1.1 and transfer the obstacle problem to another form.

Definition 2.1. Let $u \in C(\overline{\Omega})$ and $\hat{x} \in \Omega$.

(i). The second order superjet $J^{2,+}_{\Omega} u(\hat{x})$ is the set of the $(p, X) \in \mathbb{R}^n \times \mathcal{S}^n$ such that

$$u(x) \le u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2}\langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2), \text{ as } x \to \hat{x} \text{ in } \Omega,$$

where $\mathcal{S}^n$ is the set of the symmetric $n \times n$ matrices.

(ii). The second order subjet

$$J^{2,-}_{\Omega}u(\hat{x}) = \{(p, X) : \ (p, X) \in -J^{2,+}_{\Omega}(-u)(\hat{x})\}.$$

We also introduce

$$J'^{2,-}_{\Omega}u(\hat{x}) = J^{2,-}_{\Omega}u(\hat{x}) \cap (\mathbb{R}^n \times \mathcal{S}^n_+),$$

where $\mathcal{S}^n_+$ is set of positive semidefinite symmetric $n \times n$ matrices.

The following lemma is proved in [1].

Lemma 2.2. *Let $u \in C(\overline{\Omega})$. Then u is locally convex if and only if $X \geq 0$ for every $(p, X) \in J^{2,+}_{\Omega}u(x)$ and every $x \in \Omega$.*

Definition 2.3. Let $u \in C(\overline{\Omega})$ be locally convex.

(i). A function u is said to be a *viscosity solution* of (1.1), if

$$\det X \geq \psi(x, u(x), p), \quad (p, X) \in J^{2,+}_{\Omega}u(x), \ \forall x \in \Omega, \tag{2.1a}$$

$$u \leq g \quad \text{in } \Omega, \tag{2.1b}$$

$$u = \varphi \quad \text{on } \partial\Omega. \tag{2.1c}$$

(ii). A function u is said to be a viscosity subsolution (resp. supersolution) of (1.2), if for every $x \in \Omega$

$$\det X \geq (\leq)\psi(x, u(x), p), \quad (p, X) \in J^{2,+}_{\Omega}u(x) \ \big(\textit{resp. } J'^{2,-}_{\Omega}u(x)\big).$$

A function u is said to be a viscosity solution of (1.2) if it is both a viscosity subsolution and supersolution.

Note that every classical solution is a viscosity solution.

In next theorem, we only need $\psi \geq 0$ that means our equations may be degenerate.

Theorem 2.4. *(i) Assume that there exists a function $\underline{u} \in \mathcal{A}$. Then*

$$u(x) =: \sup_{v \in \mathcal{A}} v(x)$$

is still in the class $\mathcal{A}$ and satisfies

$$\det D^2u(x) = \psi(x, u(x), Du(x)), \ \ x \in E =: \{x \in \Omega : u(x) < g(x)\} \tag{2.2}$$

in viscosity sense. If $\underline{u} \in C^{0,1}(\overline{\Omega})$, then $u \in C^{0,1}(\overline{\Omega})$.

(ii) If ψ has fine property, then u is the unique function satisfying

$$\begin{aligned} \max\{u - g, -(\det D^2u - \psi(x, u, Du)\} = 0 \quad & \textit{in } \Omega, \\ u \geq \underline{u} \quad & \textit{in } \Omega, \\ u \textit{ is locally convex in } \Omega, & \\ u = \varphi \quad & \textit{on } \partial\Omega, \end{aligned} \tag{2.3}$$

in viscosity sense.

Proof. *(i)* Obviously, u is locally convex and satisfies (2.1b) and (2.1c). Equation (2.1a) follows from Lemma 4.2 in [7].

Let h be the harmonic extension of φ in Ω, that is h satisfying

$$\Delta h = 0 \text{ in } \Omega, \quad h = \varphi \text{ on } \partial\Omega. \tag{2.4}$$

Since u is locally convex, u is a viscosity subsolution to $\Delta u = 0$ in Ω. By comparison principle, $u \leq h$ on $\overline{\Omega}$. On the other hand, $\underline{u} \leq u$ by the assumption. Note that $h = u = \underline{u}$ on $\partial\Omega$, thus for any $x \in \partial\Omega$

$$\begin{aligned} D_\nu h(x) &\leq \liminf_{t\to 0^+} \frac{u(x) - u(x - t\nu)}{t} \\ &\leq \limsup_{t\to 0^+} \frac{u(x) - u(x - t\nu)}{t} \\ &\leq \limsup_{t\to 0^+} \frac{\underline{u}(x) - \underline{u}(x - t\nu)}{t}, \end{aligned}$$

where ν is the out normal to $\partial\Omega$. Hence by the convexity of u,

$$\|u\|_{C^{0,1}(\Omega)} \leq C, \tag{2.5}$$

where C depends only on $\|\underline{u}\|_{C^{0,1}(\overline{\Omega})}$, $\|\varphi\|_{C^1(\partial\Omega)}$ and Ω.

Next, we shall prove (2.2). If u fails to be a solution of

$$\det D^2 u \leq \psi(x, u, Du) \quad \text{in } E, \tag{2.6}$$

there will exist a point $x_0 \in E$ such that, we may assume $x_0 = 0$,

$$\det X > \psi(0, u(0), p) \text{ for some } (p, X) \in J_\Omega^{'2,-} u(0),$$

hence $X > 0$. Then by the continuity

$$u_{\delta,\gamma}(x) = u(0) + \delta + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle - \gamma|x|^2$$

is convex and satisfies $\det D^2 u_{\delta,\gamma} \geq \psi$ and $u_{\delta,\gamma}(x) \leq g(x)$ in $B_r = \{x : |x| < r\}$ for small $r, \delta, \gamma > 0$. Since

$$u(x) \geq u(0) + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + o(|x|^2),$$

if we choose $\delta = (r^2/8)\gamma$, then $u(x) > u_{\delta,\gamma}(x)$ for $r/2 \leq |x| \leq r$ if r is sufficiently small and then, by Lemma 4.2 in [7], the function

$$U(x) = \begin{cases} \max\{u(x), u_{\delta,\gamma}(x)\} & \text{if } |x| < r, \\ u(x) & \text{otherwise,} \end{cases}$$

is a viscosity solution to $\det D^2 u \geq \psi$. Note that $U(x)$ is locally convex, $U(x) \leq g(x)$ and $U(0) > u(0)$, this contradicts to the assumption of u. Thus (2.6) holds. Combining (2.6) and (2.1a), we complete the proof of (2.2).

(ii) From above it is easy to see that u satisfy (2.3). Let u_1, u_2 be two solutions to (2.3). Suppose there exists a point $x_0 \in \Omega$, such that $u_1(x_0) < u_2(x_0)$. Let G be a connected domain $G \subset \Omega$ containing x_0 such that

$$u_1(x) < u_2(x) \text{ in } G, \quad u_1 = u_2 \text{ on } \partial G.$$

Since $u_2 \leq g$ in Ω, $u_1 < g$ in G. Thus in viscosity sense

$$\det D^2 u_2 \geq \psi(x, u_2, Du_2) \quad \text{in } G$$

and

$$\det D^2 u_1 = \psi(x, u_1, Du_1) \quad \text{in } G.$$

By comparison principle, we have

$$u_2 \leq u_1 \quad \text{in } G,$$

this is a contradiction. We complete the proof. ∎

3. **Singular perturbation problem and $C^{1,1}$ regularity.** To establish the $C^{1,1}$ regularity for the greatest solution in Theorem 1.2, we consider the following singular perturbation problem

$$\begin{cases} \det D^2 u = e^{\beta_\varepsilon(u-g)}\psi(x,u,Du) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$\beta_\varepsilon(z) = \begin{cases} 0, & z \le 0, \\ z^3/\varepsilon, & z > 0, \end{cases}$$

and $\varepsilon \in (0,1)$.

Theorem 3.1. *Let $\psi > 0$ have fine property. Assume there exists a function $\underline{u} \in \mathcal{A}$ and $\underline{u} \in C^2(\overline{\Omega})$. Then for each $\varepsilon \in (0,1)$ there exists a unique solution $u_\varepsilon \in C^3(\overline{\Omega}) \cap C^4(\Omega)$ to (3.1) satisfying*

$$u_\varepsilon \ge \underline{u} \quad \text{in } \Omega \tag{3.2}$$

and

$$\|u_\varepsilon\|_{C^2(\overline{\Omega})} \le C, \tag{3.3}$$

where $C > 0$ is independent of ε.

Proof. Since $\underline{u} \le g$, $\underline{u} \in C^2(\overline{\Omega})$ is a subsolution to (3.1). Due to Theorem 1.1 of [9], there exists a unique solution $u_\varepsilon \in C^{3,\alpha}(\overline{\Omega})$ of (3.1) satisfying (3.2). By the interior regularity theory of elliptic equations, $u_\varepsilon \in C^4(\Omega)$. Then we only need to show the uniform estimates (3.3).

Since $\underline{u} \in \mathcal{A} \cap C^2(\overline{\Omega})$, then there exists a constant $\nu > 0$ such that

$$D^2 \underline{u} \ge \nu I \quad \text{on } \overline{\Omega}, \tag{3.4}$$

where I is the identity matrix. Let h be the harmonic extension of φ in Ω. By the maximum principle, we have

$$\underline{u} \le u_\varepsilon \le h \text{ in } \Omega, \quad \underline{u} = u_\varepsilon = h \text{ on } \partial\Omega.$$

Since u_ε is convex, we have

$$|u_\varepsilon| + |Du_\varepsilon| \le C_1 \quad \text{on } \overline{\Omega}, \tag{3.5}$$

where the constant $C_1 > 0$ depends only on Ω, n, $\|\varphi\|_{C^1(\partial\Omega)}$ and $\|\underline{u}\|_{C^1(\overline{\Omega})}$ and is independent of ε. From (3.5), there exist constants ψ_0, ψ_1 (independent of ε) such that

$$0 < \psi_0 \le \psi(x, u_\varepsilon(x), Du_\varepsilon(x)) \le \psi_1. \tag{3.6}$$

(a) Bounds for $|D^2 u_\varepsilon|$ on $\partial\Omega$. Since $g > h = u_\varepsilon$ on $\partial\Omega$ and $u_\varepsilon \le h$ in Ω, there exists a small constant $\delta > 0$ (independent of ε) such that

$$\beta_\varepsilon(u_\varepsilon - g) = 0 \quad \text{in } \Omega_\delta,$$

where $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$. Then

$$\det D^2 u_\varepsilon = \psi(x, u_\varepsilon, Du_\varepsilon) \text{ in } \Omega_\delta, \quad u_\varepsilon = \varphi \text{ on } \partial\Omega.$$

By the same procedure used to establish boundary estimates for second order derivatives in Theorem 2.1 of [9], we have

$$|D^2 u_\varepsilon| \le C_2 \quad \text{on } \partial\Omega, \tag{3.7}$$

where the constant $C_2 > 0$ depends on $\|\psi\|_{C^2(\overline{\Omega}\times[-C_1,C_1]\times[-C_1,C_1]^n)}$, $\|\varphi\|_{C^4(\partial\Omega)}$, $\|\underline{u}\|_{C^2(\overline{\Omega})}$, Ω and n, and is independent of ε.

(a) Bounds for $|D^2u_\varepsilon|$ in Ω. Firstly, we need the following lemma.

Lemma 3.2. *There exists a constant c_0 independent of ε such that*

$$0 \le \beta_\varepsilon(u_\varepsilon(x) - g(x)) \le c_0 \quad \text{in } \Omega. \tag{3.8}$$

Proof. Let $u_\varepsilon(x_0) - g(x_0) = \sup_{x\in\overline{\Omega}}(u_\varepsilon(x) - g(x))$, without loss of generality, we may suppose that $x_0 \in \Omega$. At x_0, we have $Du_\varepsilon(x_0) = Dg(x_0)$ and $D^2u_\varepsilon(x_0) \le D^2g(x_0)$, and then

$$\begin{aligned}\beta_\varepsilon(u_\varepsilon - g)(x_0) &= \log\det D^2u_\varepsilon(x_0) - \log\psi(x_0, u_\varepsilon(x_0), Du_\varepsilon(x_0))\\ &\le \log\det D^2g(x_0) - \log\psi(x_0, u_\varepsilon(x_0), Du_\varepsilon(x_0))\\ &\le \log\det D^2g(x_0) - \log\psi_0 =: c_0,\end{aligned}$$

where we have used (3.6) and the constant $c_0 > 0$ is independent of ε. Hence the lemma follows. □

To simplify the notations, we will use u instead of u_ε from now on.

Set

$$W = \max_{x\in\Omega,\ \xi\in\mathbb{S}^n}\Big\{D_{\xi\xi}u\exp\Big\{\frac{a}{2}|D(u-g)|^2 + \frac{b}{2}|x|^2\Big\}\Big\},$$

where $a,\ b$ are positive constants to be determined later. In order to establish (3.3) it suffices to derive a bound for W.

If W occurs on $\partial\Omega$, then W can be estimated via our known estimates (3.7). So we may assume W is achieved at a point $x_0 \in \Omega$ and for some unit vector $\xi \in \mathbb{S}^n$. We may suppose $\xi = e_1 = (1, 0, \cdots, 0)$, then $D_{1j}u(x_0) = 0$ for $j > 1$. By rotating the coordinates $\{x_2, \cdots, x_n\}$, we may assume $D^2u(x_0)$ is diagonal. We may also assume $D_{11}u(x_0) \ge D_{11}g(x_0)$, otherwise we are done. Let $F(D^2u) = \log\det D^2u$, we have

$$(F_{ij}) = \Big(\frac{\partial F}{\partial u_{ij}}\Big) = (D^2u)^{-1},\ \ \frac{\partial^2 F}{\partial u_{ij}\partial u_{kl}} = F_{ij,kl} = -F_{ik}F_{jl}.$$

Let L be the linearized operator at x_0

$$L = F_{ij}(D^2u(x_0))D_{ij}.$$

Since W is achieved at x_0, it follows that the function

$$h = \log D_{11}u + \frac{a}{2}|D(u-g)|^2 + \frac{b}{2}|x|^2$$

also attains its maximum at x_0 for the constants $a \ge 1$ and $b > 0$ to be determined later, and consequently

$$Dh(x_0) = 0 \text{ and } D^2h(x_0) \le 0. \tag{3.9}$$

Since $(F_{ij}(D^2u(x_0)))$ is diagonal,

$$L(h)(x_0) = (F_{ii}(D^2u(x_0)))D_{ii}h(x_0) = (D_{ii}u(x_0))^{-1}D_{ii}h(x_0) \le 0. \tag{3.10}$$

Now,

$$D_ih = \frac{D_{11i}u}{D_{11}u} + aD_k(u-g)D_{ki}(u-g) + bx_i, \tag{3.11}$$

$$\begin{aligned}D_{ii}h =& \frac{D_{11ii}u}{D_{11}u} - \frac{(D_{11i}u)^2}{(D_{11}u)^2} + \sum_k a(D_{ki}(u-g))^2\\ &+ aD_k(u-g)D_{kii}(u-g) + b.\end{aligned} \tag{3.12}$$

Rewrite equation (3.1) as

$$\log\det D^2u = \beta_\varepsilon(u-g) + f(x,u,Du),$$

where $f(x,u,Du) = \log\psi(x,u,Du)$, and differentiate it to obtain at x_0,

$$\sum_i (D_{ii}u)^{-1}D_{iik}u = D_k(\beta_\varepsilon(u-g)+f) \quad \text{for all } k, \tag{3.13}$$

$$L(D_{11}u) - \sum_{ij}\frac{(D_{1ij}u)^2}{D_{ii}uD_{jj}u} = D_{11}f + \beta'_\varepsilon(u-g)D_{11}(u-g) + \beta''_\varepsilon(u-g)(D_1(u-g))^2.$$

Since $\beta'_\varepsilon,\ \beta''_\varepsilon \geq 0$ and $D_{11}u(x_0) \geq D_{11}g(x_0)$,

$$L(D_{11}u) \geq \sum_{ij}\frac{(D_{1ij}u)^2}{D_{ii}uD_{jj}u} + D_{11}f. \tag{3.14}$$

By replacing (3.12) into (3.10) and multiplying it by $D_{11}u(x_0)$, we see that

$$\begin{aligned} 0 \geq & L(D_{11}u) - \sum_i \frac{(D_{11i}u)^2}{D_{11}uD_{ii}u} + aD_{11}u\Delta u - 2aD_{11}u\Delta g \\ & + \sum_i aD_k(u-g)\frac{D_{11}u}{D_{ii}u}D_{iik}u + \sum_i \frac{D_{11}u}{D_{ii}u}(b - aD_k(u-g)D_{iik}g). \end{aligned}$$

From (3.13) and (3.14), choosing

$$b = a\sup_{x\in\Omega}|D_k(u-g)D_{iik}g|,$$

in view of the convexity of u we infer that

$$\begin{aligned} 0 &\geq D_{11}f + a(D_{11}u)^2 - 2aD_{11}u\Delta g + aD_k(u-g)D_{11}uD_k(\beta_\varepsilon(u-g)+f) \\ &\geq \sum_j f_{p_j}D_{j11}u + (a+f_{p_1p_1})(D_{11}u)^2 + a\beta'_\varepsilon(u-g)|D(u-g)|^2D_{11}u \\ &\quad + aD_k(u-g)D_{11}uf_{p_j}D_{jk}u - Ca(1+D_{11}u). \end{aligned}$$

Since $Dh(x_0) = 0$ and (3.11), we have

$$\sum_j f_{p_j}D_{j11}u + \ aD_k(u-g)D_{11}uf_{p_j}D_{jk}u = f_{p_j}D_{11}u(aD_k(u-g)D_{kj}g - bx_j).$$

Thus

$$0 \geq (a+f_{p_1p_1})(D_{11}u)^2 - Ca(1+D_{11}u).$$

Choosing

$$a = \sup_{x\in\Omega}|f_{p_1p_1}(x,u(x),Du(x))| + 1,$$

then

$$0 \geq (D_{11}u)^2 - C(1+D_{11}u)$$

and hence (for a different C)

$$D_{11}u(x_0) \leq C,$$

which implies

$$W \leq C,$$

and hence

$$\|D^2u\|_{L^\infty(\Omega)} \leq C_3, \tag{3.15}$$

where the constant C_3 is independent of ε.

Combining (3.5), (3.7) and (3.15), we complete the proof of (3.3). □

Proof of Theorem 1.2. According to the uniformly estimates (3.3), there exists a subsequence u_{ε_k} and a function $u \in C^{1,1}(\overline{\Omega})$ such that

$$u_{\varepsilon_k} \to u \quad \text{in } C^{1,\alpha}(\overline{\Omega}),\ \forall \alpha \in (0,1),\ \text{as } \varepsilon_k \to 0.$$

Obviously, $\underline{u} \leq u$ and the inequality $u \leq g$ in Ω follows from Lemma 3.2. Then by the stable property of viscosity solution theory (see [7]), it is easy see that u is a solution of (2.3). According to *(ii)* of Theorem 2.4, u is the greatest solution of (1.1). Thus we complete the proof of Theorem 1.2. □

In fact, $u \in C^{3,\alpha}(E)$ for any $\alpha \in (0,1)$, where E as in Theorem 2.4. This follows from (i) of Theorem 2.4, Evans-Krylov estimates and Schauder estimates for nonlinear elliptic equations, see [8] or [3].

4. **Another Monge-Ampère type equations.**

We shall treat one more problem

$$\begin{cases} \max\{(u-g), -(\det(D_{ij}u - \sigma_{ij}(x)) - \psi(x))\} = 0 & \text{in } \Omega, \\ (D_{ij}u - \sigma_{ij}) \geq 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $g,\ \varphi,\ \psi$ and Ω as before, $(\sigma_{ij}(x)) \in C^2(\overline{\Omega})$ a symmetric matrix function.

Without loss of generality we can assume always that (σ_{ij}) is nonnegative definite and u is convex. The reason is that we can choose a very large number Λ, such that $\Lambda I + (\sigma_{ij})$ is nonnegative definite, then let $u = v - \Lambda|x|^2$ and solve the problem for v.

The Dirichlet problem for equation

$$\det(D_{ij}u - \sigma_{ij}) = \psi(x) \quad \text{in } \Omega, \tag{4.2}$$

has been treated by Caffarelli, Nirenberg and Spruck [6], and by Li [14] for general right hand side.

Proposition 1. *Assume Ω is strictly convex, there exists a function $\underline{u} \in C^3(\overline{\Omega})$ such that*

$$\underline{u} \leq g,\ (D_{ij}\underline{u} - \sigma_{ij}) \geq 0 \text{ and } \det(D_{ij}\underline{u} - \sigma_{ij}) \geq \psi(x) \quad \text{in } \Omega \tag{4.3}$$

and

$$\underline{u} = \varphi \quad \text{on } \partial\Omega. \tag{4.4}$$

Proof. Let u_1 be a solution to (4.2) with $u_1 = \varphi$ on $\partial\Omega$ (see [6]) and u_2 be a convex solution to equation $\det D^2 u = 1$ in Ω with $u_2 = 0$ on $\partial\Omega$. Let $u = u_1 + \lambda u_2$ with constant $\lambda > 0$, then u is a subsolution to (4.2). Since $\varphi > g$ on $\partial\Omega$, $u_2 < 0$ in Ω (see [4]) and u_2 is strictly convex, $u \leq g$ by choosing large λ. □

As before, we have

Theorem 4.1. *Let $\underline{u} \in C^2(\overline{\Omega})$ satisfying (4.3) and (4.4), then there exists unique function $u \in C^{1,1}(\overline{\Omega})$ satisfying (4.1) and $u \geq \underline{u}$ in Ω.*

When $\psi = 1,\ \varphi = 0,\ \sigma_{ij} = 0$ and Ω is strictly convex, this theorem was proved by Lee [13]. In view of Proposition 1, Theorem 4.1 is an extension of results in [13], too.

Proof of Theorem 4.1. As in section 3, we consider the singulary perturbation problem

$$\begin{cases} \det(D_{ij}u - \sigma_{ij}) = e^{\beta_\varepsilon(u-g)}\psi(x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $\beta_\varepsilon(\cdot)$ as (3.1) and $\varepsilon \in (0,1)$.

Note that $\underline{u}$ is a subsolution of (4.5). By the same approach used in [9] and [11], it follows that there exists unique solution $\underline{u} \le u_\varepsilon \in C^{3,\alpha}(\overline{\Omega})$ to (4.5) for any $\varepsilon \in (0,1)$. To complete the proof, we need to establish uniformly estimates similar to (3.3). Mimicking the procedure in the proof of Theorem 3.1, we can estimate $\|u_\varepsilon\|_{C^1(\overline{\Omega})}$ and bounds for D^2u_ε on $\partial\Omega$. To prove the bounds for D^2u_ε in Ω, we choose

$$W = \max_{x\in\Omega,\ \xi\in\mathbb{S}^n} \left\{ U_{\xi\xi} \exp\left\{ \frac{a}{2}|D(u-g)|^2 + \frac{b}{2}|x|^2 \right\} \right\},$$

where $a,\ b$ are positive constants to be determined later and

$$U_{\xi\xi}(x) = (D_{ij}u(x) - \sigma_{ij}(x))\xi_i\xi_j.$$

The rest computation is similar and we omit it here.

Once the uniformly estimates for D^2u_ε at hand, we conclude that there exists a function $u \in C^{1,1}(\overline{\Omega})$ satisfying (4.1). The uniqueness can be proved from classical comparison principle, see the proof of *(ii)* of Theorem 2.4. □

REFERENCES

[1] O. Alvarez, J.-M. Lasry and P.-L. Lions, *Convex viscosity solutions and state constraints*, J. Math. Pures Appl., **76** (1997), 265–288.

[2] J. Bao, *The obstacle problems for second order fully nonlinear elliptic equations with Neumann boundary conditions*, J. Partial Diff. Eqn., **3** (1992), 33–45.

[3] L. Caffarelli and X. Cabré, "Fully Nonlinear Elliptic Equations," Mathematical Society Colloquium Publications, 43. Amer. Math. Soc., Providence, RI, 1995.

[4] L. Caffarelli, *A Localization property of viscosity solutions to the Monge-Ampere equation and their strict convexity*, Ann. of Math., **131** (1990), 129–134.

[5] L. Caffarelli and R. McCann, *Free boundaries in optimal transport and Monge-Ampère obstacle problems*, Ann. of Math., to appear.

[6] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations I. Monge-Ampère equations*, Comm. Pure Appl. Math., **37** (1984), 369–402.

[7] M. Crandall, H. Ishii and P. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., **27** (1992), 1–67.

[8] D. Gilbarg and N. S. Trudinger, "Elliptic Partial Diiferential Equations of Second Order," Second Edition, Springer, Berlin, 1983.

[9] B. Guan, *The Dirichlet problem for Monge-Ampère equations in non-convex domains and spacelike hypersurfaces of constant Gauss curvature* , Trans. Amer. Math. Soc., **350** (1998), 4955–4971.

[10] B. Guan and Y. Y. Li, *Monge-Ampère equations on Riemannian manifolds*, J. Diff. Eqn., **132** (1996), 126–139.

[11] B. Guan and J. Spruck, *Boundary value problem on $\mathbb{S}^n$ for surfaces of constant Gauss curvature*, Ann. of Math., **138** (1993), 601–624.

[12] C. Gutiérrez, "The Monge-Ampère equation," Progress in Nonlinear Differential Equations and their Applications, 44, Birkhäuser, Boston, 2001.

[13] K. Lee, *The obstacle problem for Monge-Ampère equation*, Comm. Partial Diff. Eqn., **26** (2001), 33–42.

[14] Y. Y. Li, *Some existence results of fully nonlinear elliptic equations of Monge-Ampère type*, Comm. Pure Appl. Math., **43** (1990), 233–371.

[15] X. N. Ma, N. S. Trudinger and X-J. Wang, *Regularity of potential functions of the optimal transportation problem*, Arch. Rational Mech. Anal., **177** (2005),151–183.

[16] O. Savin, *The obstacle problem for Monge-Ampère equation*, Calc. Var. Partial Diff. Eqn., **22** (2005), 303–320.

[17] N. S. Trudinger, *The Dirichlet problem for the prescribed curvature equations*, Arch. Rational Mech. Anal., **111** (1990), 153–179.

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